The Hamiltonian Mean Field Model: Effect of Network Structure on Synchronization Dynamics

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Kuramoto Model\textsuperscript{1}

- Mutual synchronization: Coupled oscillators characterized by their phases
- All-to-all case:
  \[
  \dot{\theta}_n = \omega_n + \frac{K}{N} \sum_{m=1}^{N} \sin (\theta_m - \theta_n)
  \]
- K: coupling strength
- N: number of oscillators
- \(\theta\): phase
- \(\omega\): randomly chosen intrinsic frequency

Hamiltonian Mean Field Model\textsuperscript{2}

• Version of Kuramoto model that conserves energy

\[ H = \frac{1}{2} \sum_{n} p_{n}^{2} - \frac{K}{2N} \sum_{n,m} A_{nm} \cos (\theta_{m} - \theta_{n}) \]

\[ A_{nm} = A_{mn} \]

Kinetic Energy

Potential Energy

\[ \dot{\theta}_{n} = p_{n} \]

\[ \dot{p}_{n} = \frac{K}{N} \sum_{m=1}^{N} A_{nm} \sin (\theta_{m} - \theta_{n}) \]

Hamiltonian Mean Field Model (2)

• Link to video—
  – http://bit.ly/1YEmWAj

• This video shows the evolution of phase-space for the HMF model for a high coupling constant.
Recent work has focused on partial extensions to HMF for the network case:


In this work, we develop a more general framework.

Global order parameter

Desynchronized state (small $K$)

$$R_n e^{i\psi_n} = \frac{1}{N} \sum_{m=1}^{N} A_{nm} e^{i\theta_m}$$

Synchronized state (large $K$)

$$R = \frac{1}{\langle d \rangle} \sum_{n=1}^{N} R_n$$
Global order parameter as a function of coupling strength
Critical coupling constant $K_c$

- For initial conditions:
  - $\theta_n$ is drawn from a uniform distribution.
  - $p_n$ is drawn from a Gaussian distribution with mean $\mu$ and standard deviation $\sigma$,

we get from theory (linear stability analysis $^3$),

$$K_c = \frac{2\sigma^2 N}{\lambda}$$

- $\lambda$ is the principal eigenvalue of the adjacency matrix.

Numerical example in simulated network

We construct networks with degree distribution $P(d) = C d^{-\alpha}$.
Numerical example in simulated network (2)
Synchronized solutions

• Assuming that in the synchronized state all rotors rotate with a common frequency, and letting $r_n$ denote the time average of $R_n$, we obtain the following equations which can be solved numerically,

$$r_n = \frac{1}{N} \sum_{m=1}^{N} A_{nm} \frac{I_1 \left( \frac{K r_m}{\sigma^2} \right)}{I_0 \left( \frac{K r_m}{\sigma^2} \right)}$$

$I_1$ and $I_0$ are Bessel functions.

$$\sigma^2 = \sigma_0^2 + \frac{K}{N} \sum_{n=1}^{N} r_n \frac{I_1 \left( \frac{K r_n}{\sigma^2} \right)}{I_0 \left( \frac{K r_n}{\sigma^2} \right)}$$
Result: Theoretical vs Simulated (R vs K)

Erdös-Renyi random network: N nodes with q the probability of connecting any two nodes.
Result: Theoretical vs Simulated (R vs K)

Scale-free

\[ \langle R \rangle_t \]

- Simulated data
- Theory

\[ K \]
Result: Maximum synchrony

\[ \langle R \rangle_t \]

Increasing homogeneity
Conclusions

• Network heterogeneity impacts both the onset of synchronization ($K_c$) and the path to synchrony.

• We developed a theory for quantifying the onset of synchronization and the synchronized state for the network version of HMF.

• The maximum possible synchrony is however fairly independent of network heterogeneity.
Thanks! 😊

• References: