Synchronization and Dimensionality Reduction in Networks of Hybrid Phase Oscillators: A Perspective from Legged Locomotion

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(c) S. Revzen
Outline

- Locomotion and dimensionality reduction
- Data Driven Floquet Analysis
- Gaits as networks of synchronized oscillators
- Hybrid oscillators and synchronization
- Horses, Dogs, and Robots
Outline

- Locomotion and dimensionality reduction
- Estimating oscillator phase
- Gaits as networks of synchronized oscillators
- Legged locomotion → hybrid oscillators
- Core results for hybrid oscillator networks

Data driven models of legged locomotion

Revzen & Kvalheim
SPIE-DSS 2015
doi: 10.1117/12.2178007

(Section 1 of SPIE-DSS paper)
Locomotion: a bundle of fun

- Base space: body configuration
- Fibre: 3D position and orientation SE(3)

Periodic motions on the base space give rise to translation along a subgroup of the SE(3) fibre

*Computing Reduced Equations for Robotic Systems with Constraints and Symmetries*
Ostrowski; IEEE Tr. Robot. & Autom.; vol 15 no 1 pp 111, 1999
**Templates and Anchors**

Animals have many DOF, but move “as if” they have only a few. Animals limit pose to a behaviourally relevant family of postures.

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**TEMPLATE**

Add Degrees of Freedom (legs,joints,muscles)

**ANCHOR**

Collapse Dimensions (synergies,symmetries)

Full & Koditschek (1999)
Oscillator theory & Templates

- Provable reductions
- Data Driven tools

Phillip Holmes, Princeton

John Guckenheimer, Cornell
Dynamic Mechanical Models

Lateral Leg Spring
Schmitt and Holmes, 2000

+leg with damping, muscle model
Schmitt and Holmes, 2003

+individual legs, neuron models
Seipel, Holmes and Full, 2004

+detailed animal morphology
Schmitt, Garcia, Razo, Holmes and Full, 2004

The Dynamics of Legged Locomotion: Models, Analyses, and Challenges
Holmes, Full, Koditschek and Guckenheimer
SIAM Reviews, 2006, 48, 207-304
Templates and Time Constants

1\textsuperscript{st} template direction
2\textsuperscript{nd} template direction
non-template (pose error) direction
State space

Limit Cycle
Template with Slow recovery
Posture error, Fast recovery
Is a cockroach an oscillator?
The Counterfactual Cockroach
Modeling a Cockroach as an Oscillator

Revzen, et.al. in “Progress in Motor Control” Sternad (ed.) (2008)
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Better Phase Estimation from Data

• Need an algorithm:
  - IN: multivariate data
  - OUT: phase estimate

• Traditional method: interpolate events times

• “Phaser” (2008)
  - accurate only on cycle

Revzen & Guckenheimer, Estimating the phase of synchronized oscillators, Phys Rev E, 2008, 78, 051907
Estimation error distribution

Phaser-NG performance

Event phase

Phaser

Phaser-NG

(Collaboration with S. Wilshin, RVC; C. Scott, UMich; J.M. Guckenheimer, Cornell)
2D Performance of *Phaser-NG*

![Graphs showing 2D performance of Phaser-NG](image)
8D Performance of *Phaser-NG*
Floquet Theory (pub. 1883)

- A look at linear time periodic systems

\[ \frac{d}{dt} x = A(t)x \quad \forall t \in \mathbb{R} : A(t) = A(t + T) \]

- Solutions split:

\[ X(t) = X(T)^{\lfloor t/T \rfloor} X(t \mod T) \]

- Therefore:

\[ P(\tau) := \exp \left( -\frac{\tau}{T} \ln X(T) \right) X(\tau) \]

\[ X(t) = X(T)^{\lfloor t/T \rfloor} P(t \mod T) \]

\( P() \) a T-(demi)periodic function by construction
Floquet Theory for Oscillators

- Oscillators linearize via a periodic change of coordinates

\[
\xi(\varphi) = F(\varphi) \cdot e^{\frac{1}{2\pi}(\varphi - \theta)\Lambda} \cdot F^{-1}(\theta) \cdot \xi(\theta)
\]
Finding the Dimension of a Template

Model: Coupled Van der Pol osc.
SDE integrator

Finding the dimension of slow dynamics in a rhythmic system;
Revzen & Guckenheimer; J. Roy. Soc. Interface 2012, 9, 957-971
Time constants by noise level

Finding the dimension of slow dynamics in a rhythmic system;
Revzen & Guckenheimer; J. Roy. Soc. Interface 2012, 9, 957-971
Dimension of Cockroach Template

Finding the dimension of slow dynamics in a rhythmic system;
Revzen & Guckenheimer; J. Roy. Soc. Interface 2012, 9, 957-971
Phase-driven models

A) Lateral position (bodylengths) vs. fore-aft position (bodylengths)

B) Time (ms) vs. combined velocity (normalized)

C) Global phase estimate

D) Residual phase (radian) vs. time (ms)

- $\Delta \Phi_6$: residual phase
- $\Phi_0$: baseline (9.1 Hz taken as 0)
- $\Phi_\text{extr}$: extrapolation
- $\Phi_D$: during contact (-0.9 Hz, -0.67 at t0)
- $\Phi_1$: after hurdle (-0.1 Hz, -1.64 at t1)

$t_0 = 562$ ms
$t_1 = 830$ ms

from Revzen (2009)
Phase from one (horse) leg
Horse Gaits in terms of Phase
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Phase and multiple oscillators

- Each oscillator contributes a “circle”
- Combined (product) space is a torus
  - Tori are cubes with faces identified
  - Tori are “flat” and allow for “translation”
Dog on flat terrain

Used relative phases for Caesar

Ideal trot is green circled point (s)

Ideal walk ~ red circle

[\pi/2 \pi 3\pi/4]

pace

[\pi \pi \pi]

[0 \pi 0]
Dog on rough terrain

Used relative phases for Caesar

Ideal walk

$[\pi/2, \pi, 3\pi/4]$
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Hybrid System as Linked Domains

- State-space is disjoint union of **domains**
- System “leaves” a domain at a **guard**
- Carried by **reset** map into new domain

**NOTE:** dimension can change

**Burden, Revzen & Sastry, “Model Reduction Near Periodic Orbits of Hybrid Dynamical Systems” IEEE TAC 2015**
RESULT: has invariant sub "manifold"

- Hybrid oscillators can have finite time (irreversible) transients
- Collapse to lower dimensional dynamics

(b) $H|_M = (M, F|_M, G \cap M, R|_{G \cap M})$

Burden, Revzen & Sastry, "Model Reduction Near Periodic Orbits of Hybrid Dynamical Systems" IEEE TAC 2015
RESULT: the rest is “smoothable”

- Can be “stitched together” into a smooth manifold

No uniquely hybrid behaviors exist!
(asymptotically)

Burden, Revzen & Sastry, “Model Reduction Near Periodic Orbits of Hybrid Dynamical Systems” IEEE TAC 2015
Isolated Transitions Results

Finite time arrival on an invariant sub-manifold

**Theorem 1** (Exact Reduction). Let $\gamma$ be a periodic orbit that undergoes isolated transitions in a hybrid dynamical system $H = (D, F, G, R)$, $P : U \to \Sigma$ a Poincaré map for $\gamma$, $m = \min_j \dim D_j$, and suppose there exists a neighborhood $V \subset U$ of $\{\xi\} = \gamma \cap \Sigma$ and $r \in \mathbb{N}$ such that $\text{rank } DP^m(x) = r$ for all $x \in V$. Then there exists an $(r + 1)$-dimensional hybrid embedded submanifold $M \subset D$ and a hybrid open set $W \subset D$ for which $\gamma \subset M \cap W$ and trajectories starting in $W$ contract to $M$ in finite time.

“Generic”: nearby systems are super-exponential

**Theorem 2** (Approximate Reduction). Let $\gamma$ be an exponentially stable periodic orbit undergoing isolated transitions in a hybrid dynamical system $H = (D, F, G, R)$, $P : U \to \Sigma$ a Poincaré map for $\gamma$ with fixed point $\{\xi\} = \gamma \cap \Sigma$, $m = \min_j \dim D_j$, and $r = \text{rank } DP^m(\xi)$. Then there exists an $(r + 1)$-dimensional hybrid embedded submanifold $M \subset D$ such that for any $\varepsilon > 0$ there exists a hybrid open set $W^\varepsilon \subset D$ for which $\gamma \subset M \cap W^\varepsilon$ and the distance from trajectories starting in $W^\varepsilon$ to $M$ contracts by $\varepsilon$ each cycle.

Sub-manifold is “smoothable”

**Theorem 3** (Smoothing). Let $H = (M, F, G, R)$ be a hybrid dynamical system with $M = \bigsqcup_{j \in J} M_j$. Suppose $\dim M_j = n$ for all $j \in J$, $R(G) \subset \partial M$, $\partial M = G \bigsqcup R(G)$, $R$ is a hybrid diffeomorphism onto its image, and $F$ is inward-pointing along $R(G)$. Then the topological quotient $\widetilde{M} = \frac{M}{G \circ R(G)}$ may be endowed with the structure of a smooth manifold such that:

1) the quotient projection $\pi : M \to \widetilde{M}$ restricts to a smooth embedding $\pi|_{M_j} : M_j \to \widetilde{M}$ for each $j \in J$;

2) there is a smooth vector field $\tilde{F} \in \mathcal{X}(\widetilde{M})$ such that any execution $x : T \to M$ of $H$ descends to an integral curve of $\tilde{F}$ on $\widetilde{M}$ via $\pi : M \to \widetilde{M}$:

$$\forall t \in T : \frac{d}{dt} \pi \circ x(t) = \tilde{F} (\pi \circ x(t)).$$

(c) S. Revzen
What do these gaits have in common?

(from TED talk, R.J.Full)

(Section 3 of SPIE-DSS paper)
Hybrid Systems as Piecewise Smooth

- State space is partitioned into **domains**
- Domain boundaries are codim 1 manifolds
- Flow is smooth in each domain
- We define:
  - *Event-Selected Systems*
  - Vector field monotone & transverse to guards

(Section 3 of SPIE-DSS paper)
Simultaneous Hybrid Transitions

Touchdown #1

Multiple Contact

Touchdown #2

(Section 3 of SPIE-DSS paper)
Multiple Contact Gaits

- Rewrite with respect to touchdown times

- Domains $D_q$ are now indexed by $\{-1,+1\}^d$
Multiple Contact Gaits

- Whatever is new about dynamics appears in (nearly) piecewise constant systems
First order approximations

\[ f : \mathbb{R}^d \rightarrow \mathbb{R}^d \text{ is directionally differentiable at } x \]

if there exists \( Df(x) : \mathbb{R}^d \rightarrow \mathbb{R}^d \) for which:

\[ \forall h \in \mathbb{R}^d : \lim_{s \downarrow 0} \frac{1}{s} \| f(x + sh) - (f(x) + Df(x; sh)) \| = 0 \]
First order approximations

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A continuous “first order model”

\( Df(x) : \mathbb{R}^d \to \mathbb{R}^d \) is a **Bouligand derivative** if:

\[ \lim_{h \to 0} \frac{1}{\|h\|} \| f(x + h) - (f(x) + Df(x; h)) \| = 0 \]
Intersecting Transitions: Key Results (1)

A unique PC$^r$ flow exists and is B-differentiable

**Theorem 1.** Suppose the vector field $F : D \rightarrow TD$ is event-selected $C^r$ at $\rho \in D$. Then there exists a flow $\phi : \mathcal{F} \rightarrow D$ for $F$ over a flow domain $\mathcal{F} \subset \mathbb{R} \times D$ containing $(0, \rho)$ such that $\phi \in PC^r(\mathcal{F}, D)$ and

$$\forall (t, x) \in \mathcal{F} : \phi(t, x) = x + \int_0^t F(\phi(s, x)) \, ds.$$ 

**Impact times are defined and PC$^r$**

**Theorem 2.** Suppose the vector field $F : D \rightarrow TD$ is event-selected $C^r$ at $\rho \in D$. If $\sigma \in C^r(U, \mathbb{R})$ is an event function for $F$ on an open neighborhood $\rho \in U \subset D$, then there exists an open neighborhood $\rho \in V \subset D$ and piecewise-differentiable function $\mu \in PC^r(V, \mathbb{R})$ such that

$$\forall x \in V : \sigma \circ \phi(\mu(x), x) = \sigma(\rho)$$

where $\phi \in PC^r(\mathcal{F}, D)$ is a flow for $F$ and $(0, \rho) \in \mathcal{F}$.

**Flows are piecewise smooth conjugate to flow boxes**
The dynamics are structurally stable in the product smooth function topology

Theorem 3. Let $F \in C^r \left( \bigsqcup_{b \in B_n} D, \bigsqcup_{b \in B_n} TD \right)$, $h \in C^r(D, \mathbb{R}^m)$ determine an event-selected $C^r$ vector field at $\rho \in D$, $r \geq 2$. Then for all $\varepsilon > 0$ there exists $\delta > 0$ such that for all $\tilde{F} \in B_\delta^{C^r}(F)$:

(a) pairing $h$ with the perturbed vector field $\tilde{F}$ determines an event-selected $C^r$ vector field at $\rho$;

(b) the perturbed flow $\tilde{\phi} : \tilde{F} \rightarrow D$ obtained by applying Theorem 1 to this perturbed vector field satisfies $\tilde{\phi} \in B_\varepsilon^{C^0}(\phi)$ on $\tilde{F} \cap \mathcal{F}$ and $(0, \rho) \in \tilde{F} \cap \mathcal{F}$;

Theorem 4. Let $F \in C^r \left( \bigsqcup_{b \in B_n} D, \bigsqcup_{b \in B_n} TD \right)$, $h \in C^r(D, \mathbb{R}^d)$ determine an event-selected $C^r$ vector field at $\rho \in D$ and suppose $Dh(\rho)$ is invertible and $r \geq 1$. Then for all $\varepsilon > 0$ sufficiently small there exists $\delta > 0$ such that for all $\tilde{F} \in B_\delta^{C^r}(F)$, $\tilde{h} \in B_\delta^{C^r}(h)$:

(a) there exists a unique $\tilde{\rho} \in B_\delta(\rho)$ such that $\tilde{h}(\tilde{\rho}) = 0$ and $\tilde{h}(x) \neq 0$ for all $x \in B_\delta(\rho) \setminus \{\tilde{\rho}\}$;

(b) pairing $\tilde{h}$ with the perturbed vector field $\tilde{F}$ determines an event-selected $C^r$ vector field at $\tilde{\rho}$;

(c) the perturbed flow yielded by Theorem 1, $\tilde{\phi} : \tilde{F} \rightarrow D$, satisfies $\tilde{\phi} \in B_\varepsilon^{C^0}(\phi)$ on $\tilde{F} \cap \mathcal{F} \neq \emptyset$;
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• **Horses, Dogs, and Robots**
Independent touchdown, Synchronized liftoff
Horse trotting
Horses Local Lyapunov Exponent

![Graph showing the slope of section maps vs phase with a phase lag highlighted.]
Dog trotting
Dog Local Lyapunov Exponent

The diagram shows the slope of section maps as a function of phase. The median is highlighted in magenta, and the phase lag is indicated by the blue bar. The x-axis represents the phase, ranging from 0.0 to 1.0, and the y-axis represents the slope of the section maps, ranging from 0.4 to 1.6.
Comparison Horses vs. Dogs

![Graph comparing slope of section maps for Horses and Dogs over different phases with a phase lag indicated.](image-url)
Synchronizing a tripod

(Section 3 of SPIE-DSS paper)
Controlling a RHex

- XRL robot has 6 legs
  - Gait is alternating tripods of support
  - Need to synchronize tripods in antiphase
Piecewise Constant Control
Conclusions & Thanks

- **Oscillator networks** can model animal motion
- **We have tools** to analyze such system models
- Event Selected hybrid oscillator networks:
  - Exhibit **new forms of robust stability**
  - **Do not** have new bifurcations / qualitative dynamics

http://purl.org/sburden/ECr-Yields-PCr
http://purl.org/sburden/Hybrid-Oscillator-Reduction